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SPHERICALLY SYMMETRIC AND ROTATING WORMHOLES PRODUCED BY LIGHTLIKE BRANES

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Lightlike p-branes (LL-branes) with dynamical (variable) tension allow simple and elegant Polyakov-type and dual to it Nambu–Goto-like worldvolume action formulations. Here we first briefly describe the dynamics of LL-branes as test objects in various physically interesting gravitational backgrounds of black hole type, including rotating ones. Next we show that LL-branes are the appropriate gravitational sources that provide proper matter energy–momentum tensors in the Einstein equations of motion needed to generate traversable wormhole solutions, in particular, self-consistent cylindrical rotating wormholes, with the LL-branes occupying their throats. Here a major role is being played by the dynamical LL-brane tension which turns out to be negative but may be of arbitrary small magnitude. As a particular solution we obtain traversable wormhole with Schwarzschild geometry generated by a LL-brane positioned at the wormhole throat, which represents the correct consistent realization of the original Einstein–Rosen "bridge" manifold.

Keywords: Traversable wormholes; non-Nambu–Goto lightlike branes; dynamical brane tension; black hole's horizon "straddling".

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1. Introduction

Lightlike branes (LL-branes) are very interesting dynamical systems which play an important role in the description of various physically important phenomena in general relativity, such as: (i) impulsive lightlike signals arising in cataclysmic astrophysical events;[1] (ii) the "membrane paradigm"[2] of black hole physics; (iii) the thin-wall approach to domain walls coupled to gravity.[3–5]

More recently, LL-branes became significant also in the context of modern nonperturbative string theory, in particular, as the so-called H-branes describing quantum horizons (black hole and cosmological),[6] as Penrose limits of baryonic D(= Dirichlet) branes,[7] etc. (see also Refs. 8–10).

In the original papers[3–5] LL-branes in the context of gravity and cosmology have been extensively studied from a phenomenological point of view, i.e. by introducing them without specifying the Lagrangian dynamics from which they may originate.[a] On the other hand, we have proposed in a series of recent papers[12–19] a new class of concise Lagrangian actions, providing a derivation from first principles of the LL-brane dynamics.

There are several characteristic features of LL-branes which drastically distinguish them from ordinary Nambu–Goto branes:

(i) They describe intrinsically lightlike modes, whereas Nambu–Goto branes describe massive ones.
(ii) The tension of the LL-brane arises as an *additional dynamical degree of freedom*, whereas Nambu–Goto brane tension is a given *ad hoc* constant. The latter characteristic feature significantly distinguishes our LL-brane models from the previously proposed *tensionless* p-branes (for a review, see Ref. 20) which rather resemble a p-dimensional continuous distribution of massless point-particles.
(iii) Consistency of LL-brane dynamics in a spherically or axially symmetric gravitational background of codimension one requires the presence of an event horizon which is automatically occupied by the LL-brane ("horizon straddling" according to the terminology of Ref. 4).
(iv) When the LL-brane moves as a *test* brane in spherically or axially symmetric gravitational backgrounds its dynamical tension exhibits exponential "inflation/deflation" time behavior (Refs. 16, 17 and Eqs. (33), (44) and (54) below) — an effect similar to the "mass inflation" effect around black hole horizons.[21,22]

In the present paper we will explore the novel possibility of employing LL-branes as natural self-consistent gravitational sources for wormhole space–times, in other words, generating wormhole solutions in self-consistent bulk gravity-matter systems coupled to LL-branes through dynamically derived worldvolume LL-brane stress–energy tensors. For a review of wormhole space–times, see Refs. 23–26.

The possibility of a "wormhole space–time" was first hinted at in the work of Einstein and Rosen,[27] where they considered matching at the horizon of two

[a] In a more recent paper[11] brane actions in terms of their pertinent extrinsic geometry have been proposed which generically describe non-lightlike branes, whereas the lightlike branes are treated as a limiting case.

identical copies of the exterior Schwarzschild space–time region (subsequently called Einstein–Rosen "bridge"). The original Einstein–Rosen "bridge" manifold appears as a particular case of the construction of spherically symmetric wormholes produced by LL-branes as gravitational sources (Refs. 18, 19 and Sec. 4 below). The main lesson here is that consistency of Einstein equations of motion yielding the original Einstein–Rosen "bridge" as well-defined solution necessarily requires the presence of LL-brane energy–momentum tensor as a source on the right-hand side. Thus, the introduction of LL-brane coupling to gravity brings the original Einstein–Rosen construction in Ref. 27 to a consistent completion (see the Appendix for details).

Let us particularly emphasize that here and in what follows we consider the Einstein–Rosen "bridge" in its original formulation in Ref. 27 as a four-dimensional space–time manifold consisting of two copies of the exterior Schwarzschild space–time region matched along the horizon.[b]

A more complicated example of a spherically symmetric wormhole with Reissner–Nordström geometry has also been presented in Refs. 18 and 19, where two copies of the outer Reissner–Nordström space–time region are matched via LL-brane along what used to be the outer horizon of the full Reissner–Nordström manifold (see also Sec. 4 below). In this way we obtain a wormhole solution which combines the features of the Einstein–Rosen "bridge" on the one hand (with wormhole throat at horizon), and the features of Misner–Wheeler wormholes,[31] i.e. exhibiting the so-called "charge without charge" phenomenon,[c] on the other hand.

In the present paper the results of Refs. 18 and 19 will be extended to the case of rotating (and charged) wormholes. Namely, we will construct rotating cylindrically symmetric wormhole solutions by matching two copies of the outer region of rotating cylindrically symmetric (charged) black hole via rotating LL-branes sitting at the wormhole throat which in this case is the outer horizon of the corresponding rotating black hole. Let us stress again that in doing so we will be solving Einstein equations of motion systematically derived from a well-defined action principle, i.e. a Lagrangian action describing bulk gravity–matter system coupled to a LL-brane, so that the energy–momentum tensor on the r.h.s. of Einstein equations

[b]The nomenclature of "Einstein–Rosen bridge" in several standard textbooks (e.g. Ref. 28) uses the Kruskal–Szekeres manifold. The latter notion of "Einstein–Rosen bridge" is not equivalent to the original construction in Ref. 27. Namely, the two regions in Kruskal–Szekeres space–time corresponding to the outer Schwarzschild space–time region ($r > 2m$) and labeled (I) and (III) in Ref. 28 are generally *disconnected* and share only a two-sphere (the angular part) as a common border ($U = 0$, $V = 0$ in Kruskal–Szekeres coordinates), whereas in the original Einstein–Rosen "bridge" construction the boundary between the two identical copies of the outer Schwarzschild space–time region ($r > 2m$) is a three-dimensional hypersurface ($r = 2m$).

[c]Misner and Wheeler[31] realized that wormholes connecting two asymptotically flat space times provide the possibility of "charge without charge," i.e. electromagnetically nontrivial solutions where the lines of force of the electric field flow from one universe to the other without a source and giving the impression of being positively charged in one universe and negatively charged in the other universe.

will contain as a crucial piece the explicit worldvolume stress–energy tensor of the LL-brane given by the LL-brane worldvolume action.

In Sec. 2 of the present paper we briefly review our construction of LL-brane worldvolume actions for *arbitrary* worldvolume dimensions.

In Sec. 3 we discuss the properties of LL-brane dynamics as *test* branes moving in generic spherically or axially symmetric gravitational backgrounds. In the present paper we concentrate on the special case of codimension one LL-branes. Here consistency of the LL-brane dynamics dictates that the bulk space–time must possess an event horizon which is automatically occupied by the LL-brane ("horizon straddling"). In the case of rotating black hole backgrounds the test LL-brane rotates along with the rotation of the horizon. Also, similarly to the nonrotating case we find exponential "inflation/deflation" of the test LL-brane's dynamical tension.

In Sec. 4 we consider self-consistent systems of bulk gravity and matter interacting with LL-branes. We present the explicit construction of wormhole solutions to the Einstein equations with spherically symmetric or rotating cylindrically symmetric geometry, generated through the pertinent LL-brane energy–momentum tensor.

In Sec. 5 we briefly describe the traversability of the newly found rotating cylindrical wormholes.

In the Appendix we first show that the Einstein–Rosen "bridge" solution in terms of the original coordinates introduced in Ref. 27 *does not* satisfy the vacuum Einstein equations due to an ill-defined δ-function contribution at the throat appearing on the r.h.s. — a would-be "thin shell" matter energy–momentum tensor. Then we show how our present construction of wormhole solutions via LL-branes at their throats resolves the above problem and furnishes a satisfactory completion of the original construction of Einstein–Rosen "bridge."[27] In other words, the fully consistent formulation of the original Einstein–Rosen "bridge" manifold as two identical copies of the exterior Schwarzschild space–time region matched along the horizon must include a gravity coupling to a LL-brane, which produces the proper surface stress–energy tensor (derived from a well-defined worldvolume Lagrangian) necessary for the "bridge" metric to satisfy the pertinent Einstein equations everywhere, including at the throat.

2. Lightlike Branes: Worldvolume Action Formulations

In a series of previous papers[12–19] we proposed manifestly reparametrization invariant worldvolume actions describing intrinsically lightlike p-branes for any worldvolume dimension $(p+1)$:

$$S = -\int d^{p+1}\sigma\, \Phi \left[\frac{1}{2}\gamma^{ab}\partial_a X^{\mu}\partial_b X^{\nu} G_{\mu\nu}(X) - L(F^2)\right] . \tag{1}$$

Here the following notions and notations are used:

- Φ is alternative non-Riemannian integration measure density (volume form) on the p-brane worldvolume manifold:

$$\Phi \equiv \frac{1}{(p+1)!}\,\varepsilon^{a_1\cdots a_{p+1}} H_{a_1\cdots a_{p+1}}(B)\,,$$
$$H_{a_1\cdots a_{p+1}}(B) = (p+1)\partial_{[a_1} B_{a_2\cdots a_{p+1}]} \tag{2}$$

instead of the usual $\sqrt{-\gamma}$. Here γ_{ab} $(a,b = 0,1,\ldots,p)$ indicates the intrinsic Riemannian metric on the worldvolume, and $\gamma = \det \|\gamma_{ab}\|$. $H_{a_1\cdots a_{p+1}}(B)$ denotes the field-strength of an auxiliary worldvolume antisymmetric tensor gauge field $B_{a_1\cdots a_p}$ of rank p. As a special case one can build $H_{a_1\cdots a_{p+1}}$ in terms of $p+1$ auxiliary worldvolume scalar fields $\left\{\varphi^I\right\}_{I=1}^{p+1}$:

$$\Phi \equiv \frac{1}{(p+1)!}\,\varepsilon_{I_1\cdots I_{p+1}}\varepsilon^{a_1\cdots a_{p+1}}\partial_{a_1}\varphi^{I_1}\cdots\partial_{a_{p+1}}\varphi^{I_{p+1}}\,. \tag{3}$$

Note that γ_{ab} is *independent* of the auxiliary worldvolume fields $B_{a_1\cdots a_p}$ or φ^I. The alternative non-Riemannian volume form (2) has been first introduced in the context of modified standard (non-lightlike) string and p-brane models in Refs. 29 and 30.

- $X^\mu(\sigma)$ are the p-brane embedding coordinates in the bulk D-dimensional space–time with bulk Riemannian metric $G_{\mu\nu}(X)$ with $\mu,\ \nu = 0,1,\ldots,D-1$; $(\sigma) \equiv (\sigma^0 \equiv \tau, \sigma^i)$ with $i = 1,\ldots,p$; $\partial_a \equiv \frac{\partial}{\partial\sigma^a}$.
- g_{ab} is the induced metric:

$$g_{ab} \equiv \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X)\,, \tag{4}$$

which becomes *singular* on-shell (manifestation of the lightlike nature, cf. Eq. (10) below).

- $L(F^2)$ is the Lagrangian density of another auxiliary $(p-1)$-rank antisymmetric tensor gauge field $A_{a_1\cdots a_{p-1}}$ on the worldvolume with p-rank field-strength and its dual:

$$F_{a_1\cdots a_p} = p\partial_{[a_1} A_{a_2\cdots a_p]}\,, \quad F^{*a} = \frac{1}{p!}\,\frac{\varepsilon^{aa_1\cdots a_p}}{\sqrt{-\gamma}}\,F_{a_1\cdots a_p}\,. \tag{5}$$

$L(F^2)$ is *arbitrary* function of F^2 with the short-hand notation:

$$F^2 \equiv F_{a_1\cdots a_p} F_{b_1\cdots b_p}\gamma^{a_1 b_1}\cdots\gamma^{a_p b_p}\,. \tag{6}$$

Let us note the simple identity:

$$F_{a_1\cdots a_{p-1} b} F^{*b} = 0\,, \tag{7}$$

which will play a crucial role in the sequel.

Rewriting the action (1) in the following equivalent form:

$$S = -\int d^{p+1}\sigma\,\chi\sqrt{-\gamma}\left[\frac{1}{2}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) - L(F^2)\right], \quad \chi \equiv \frac{\Phi}{\sqrt{-\gamma}} \tag{8}$$

with Φ the same as in (2), we find that the composite field χ plays the role of a *dynamical* (*variable*) *brane tension*. Let us note that the notion of dynamical brane tension has previously appeared in different contexts in Refs. 32–34.

Now let us consider the equations of motion corresponding to (1) w.r.t. $B_{a_1\cdots a_p}$:

$$\partial_a\left[\frac{1}{2}\gamma^{cd}g_{cd} - L(F^2)\right] = 0 \to \frac{1}{2}\gamma^{cd}g_{cd} - L(F^2) = M\,, \tag{9}$$

where M is an arbitrary integration constant. The equations of motion w.r.t. γ^{ab} read:

$$\frac{1}{2}g_{ab} - F^2 L'(F^2)\left[\gamma_{ab} - \frac{F^*_a F^*_b}{F^{*\,2}}\right] = 0\,, \tag{10}$$

where $F^{*\,a}$ is the dual field strength (5). In deriving (10) we made an essential use of the identity (7).

Before proceeding, let us mention that both the auxiliary worldvolume field $B_{a_1\cdots a_p}$ entering the non-Riemannian integration measure density (2), as well as the intrinsic worldvolume metric γ_{ab} are *nondynamical* degrees of freedom in the action (1), or equivalently, in (8). Indeed, there are no (time-)derivatives w.r.t. γ_{ab}, whereas the action (1) (or (8)) is *linear* w.r.t. the velocities $\partial_0 B_{a_1\cdots a_p}$. Thus, (1) is a constrained dynamical system, i.e. a system with gauge symmetries including the gauge symmetry under worldvolume reparametrizations, and both Eqs. (9) and (10) are in fact *nondynamical constraint* equations (no second-order time derivatives present). Their meaning as constraint equations is best understood within the framework of the Hamiltonian formalism for the action (1). The latter can be developed in strict analogy with the Hamiltonian formalism for a simpler class of modified *non-lightlike* p-brane models based on the alternative non-Riemannian integration measure density (2), which was previously proposed in Ref. 35 (for details, we refer to Secs. 2 and 3 of Ref. 35). In particular, Eq. (10) can be viewed as p-brane analogues of the string Virasoro constraints.

There are two important consequences of Eqs. (9), (10). Taking the trace in (10) and comparing with (9) implies the following crucial relation for the Lagrangian function $L(F^2)$:

$$L(F^2) - pF^2L'(F^2) + M = 0\,, \tag{11}$$

which determines F^2 (6) on-shell as certain function of the integration constant M (9), i.e.

$$F^2 = F^2(M) = \text{const}\,. \tag{12}$$

The second and most profound consequence of Eqs. (10) is that the induced metric (4) on the worldvolume of the p-brane model (1) is *singular* on-shell (as opposed to the induced metric in the case of ordinary Nambu–Goto branes):

$$g_{ab}F^{*\,b} = 0\,, \tag{13}$$

i.e. the tangent vector to the worldvolume $F^{*\,a}\partial_a X^\mu$ is *lightlike* w.r.t. metric of the embedding space–time. Thus, we arrive at the following important conclusion: every point on the surface of the p-brane (1) moves with the speed of light in a time-evolution along the vector-field F^{*a} which justifies the name LL-brane (Lightlike-brane) model for (1).

Remark. Let us stress the importance of introducing the alternative non-Riemannian integration measure density in the form (2). If we would have started with worldvolume LL-brane action in the form (8) where the tension χ would be an *elementary* scalar field (instead of being a composite one — a ratio of two scalar densities as in the second relaton in (8)), then variation w.r.t. χ would produce second Eq. (9) with M identically zero. This in turn by virtue of the constraint (11) (with $M = 0$) would require the Lagrangian $L(F^2)$ to assume the special fractional function form $L(F^2) = (F^2)^{1/p}$. In this special case the action (8) with elementary field χ becomes in addition manifestly invariant under *Weyl* (*conformal*) *symmetry*: $\gamma_{ab} \to \gamma'_{ab} = \rho\gamma_{ab}$, $\chi \to \chi' = \rho^{\frac{1-p}{2}}\chi$. This special case of Weyl-conformally invariant LL-branes has been discussed in our older papers.[12,13]

Before proceeding let us point out that we can add[12–15] to the LL-brane action (1) natural couplings to bulk Maxwell and Kalb–Ramond gauge fields. The latter do not affect Eqs. (9) and (10), so that the conclusions about on-shell constancy of F^2 (12) and the lightlike nature (13) of the p-branes under consideration remain unchanged.

Further, the equations of motion w.r.t. worldvolume gauge field $A_{a_1\cdots a_{p-1}}$ (with χ as defined in (8) and accounting for the constraint (12)) read:

$$\partial_{[a}\big(F^*_{b]}\chi\big) = 0\,. \tag{14}$$

They allow us to introduce the dual "gauge" potential u:

$$F^*_a = \text{const}\,\frac{1}{\chi}\partial_a u\,, \tag{15}$$

enabling us to rewrite Eq. (10) (the lightlike constraint) in terms of the dual potential u in the form:

$$\gamma_{ab} = \frac{1}{2a_0}\,g_{ab} - \frac{(2a_0)^{p-2}}{\chi^2}\,\partial_a u\partial_b u\,, \quad a_0 \equiv F^2L'(F^2)\big|_{F^2=F^2(M)} = \text{const} \tag{16}$$

($L'(F^2)$ denotes derivative of $L(F^2)$ w.r.t. the argument F^2). From (15) and (12) we obtain the relation:

$$\chi^2 = -(2a_0)^{p-2}\gamma^{ab}\partial_a u\partial_b u\,, \tag{17}$$

and the Bianchi identity $\nabla_a F^{*a} = 0$ becomes:

$$\partial_a \left(\frac{1}{\chi} \sqrt{-\gamma}\gamma^{ab}\partial_b u \right) = 0 \,. \tag{18}$$

Finally, the X^μ equations of motion produced by the (1) read:

$$\partial_a \left(\chi\sqrt{-\gamma}\gamma^{ab}\partial_b X^\mu \right) + \chi\sqrt{-\gamma}\gamma^{ab}\partial_a X^\nu \partial_b X^\lambda \Gamma^\mu_{\nu\lambda}(X) = 0 \,, \tag{19}$$

where $\Gamma^\mu_{\nu\lambda} = \frac{1}{2} G^{\mu\kappa}(\partial_\nu G_{\kappa\lambda} + \partial_\lambda G_{\kappa\nu} - \partial_\kappa G_{\nu\lambda})$ is the Christoffel connection for the external metric.

Now it is straightforward to prove that the system of Eqs. (17)–(19) for (X^μ, u, χ), which are equivalent to the equations of motion (9)–(14), (19) resulting from the original Polyakov-type LL-brane action (1), can be equivalently derived from the following *dual* Nambu–Goto-type worldvolume action:

$$S_{\mathrm{NG}} = -\int d^{p+1}\sigma\, T \sqrt{-\det \left\| g_{ab} - \frac{1}{T^2}\partial_a u \partial_b u \right\|} \,. \tag{20}$$

Here g_{ab} is the induced metric (4); T is *dynamical* tension simply related to the dynamical tension χ from the Polyakov-type formulation (8) as $T^2 = \frac{\chi^2}{(2a_0)^{p-1}}$ with a_0 — same constant as in (16).

In what follows we will consider the initial Polyakov-type form (1) of the LL-brane worldvolume action. Worldvolume reparametrization invariance allows to introduce the standard synchronous gauge-fixing conditions:

$$\gamma^{0i} = 0 \quad (i = 1, \ldots, p) \,, \qquad \gamma^{00} = -1 \,. \tag{21}$$

Also, we will use a natural ansatz for the "electric" part of the auxiliary worldvolume gauge field-strength:

$$F^{*i} = 0 \quad (i = 1, \ldots, p) \,, \quad \text{i.e. } F_{0i_1\cdots i_{p-1}} = 0 \,, \tag{22}$$

meaning that we choose the lightlike direction in Eq. (13) to coincide with the brane proper-time direction on the worldvolume ($F^{*a}\partial_a \sim \partial_\tau$). The Bianchi identity ($\nabla_a F^{*a} = 0$) together with (21), (22) and the definition for the dual field-strength in (5) imply:

$$\partial_0 \gamma^{(p)} = 0 \quad \text{where} \quad \gamma^{(p)} \equiv \det \|\gamma_{ij}\| \,. \tag{23}$$

Then LL-brane equations of motion acquire the form (recall definition of g_{ab} (4)):

$$g_{00} \equiv \dot{X}^\mu G_{\mu\nu} \dot{X}^\nu = 0 \,, \quad g_{0i} = 0 \,, \quad g_{ij} - 2a_0\, \gamma_{ij} = 0 \tag{24}$$

(the latter are analogs of Virasoro constraints), where the M-dependent constant a_0 (the same as in (16)) must be strictly positive;

$$\partial_i \chi = 0 \quad \text{(remnant of Eq. (14))} \,; \tag{25}$$

$$-\sqrt{\gamma^{(p)}}\partial_0(\chi\partial_0 X^\mu) + \partial_i\left(\chi\sqrt{\gamma^{(p)}}\gamma^{ij}\partial_j X^\mu\right) + \chi\sqrt{\gamma^{(p)}}(-\partial_0 X^\nu \partial_0 X^\lambda + \gamma^{kl}\partial_k X^\nu \partial_l X^\lambda)\Gamma^\mu_{\nu\lambda} = 0 \,. \tag{26}$$

3. Lightlike Test-Branes in Spherically and Axially Symmetric Gravitational Backgrounds

First, let us consider codimension one LL-brane moving in a general spherically symmetric background:

$$ds^2 = -A(t,r)(dt)^2 + B(t,r)(dr)^2 + C(t,r)h_{ij}(\boldsymbol{\theta})d\theta^i\, d\theta^j\,, \tag{27}$$

i.e. $D = (p+1)+1$, with the simplest nontrivial ansatz for the LL-brane embedding coordinates $X^\mu(\sigma)$:

$$t = \tau \equiv \sigma^0\,, \quad r = r(\tau)\,, \quad \theta^i = \sigma^i \;(i = 1,\ldots,p)\,. \tag{28}$$

The LL-brane equations of motion (16)–(19), taking into account (21) and (22), acquire the form:

$$-A + B\dot{r}^2 = 0\,, \quad \text{i.e.} \quad \dot{r} = \pm\sqrt{\frac{A}{B}}\,, \quad \partial_t C + \dot{r}\partial_r C = 0\,, \tag{29}$$

$$\partial_\tau \chi + \chi \left[\partial_t \ln\sqrt{AB} \pm \frac{1}{\sqrt{AB}}\left(\partial_r A + p\, a_0 \partial_r \ln C\right)\right]_{r=r(\tau)} = 0\,, \tag{30}$$

where a_0 is the same constant appearing in (16). In particular, we are interested in static spherically symmetric metrics in standard coordinates:

$$ds^2 = -A(r)(dt)^2 + A^{-1}(r)(dr)^2 + r^2 h_{ij}(\boldsymbol{\theta})d\theta^i\, d\theta^j \tag{31}$$

for which Eqs. (29) yield:

$$\dot{r} = 0\,, \quad \text{i.e.} \quad r(\tau) = r_0 = \text{const}\,, \quad A(r_0) = 0\,. \tag{32}$$

Further, Eq. (30) implies for the dynamical tension:

$$\chi(\tau) = \chi_0 \exp\left\{\mp\tau\left(\partial_r A\Big|_{r=r_0} + \frac{2pa_0}{r_0}\right)\right\}, \quad \chi_0 = \text{const}\,. \tag{33}$$

Thus, we find a time-asymmetric solution for the dynamical brane tension which (upon appropriate choice of the signs ($\mp$) depending on the sign of the constant factor in the exponent on the r.h.s. of (33)) *exponentially "inflates" or "deflates"* for large times (for details we refer to Refs. 16 and 17). This phenomenon is an analog of the "mass inflation" effect around black hole horizons.[21,22]

Next, let us consider ($D = 4$)-dimensional Kerr–Newman background metric in the standard Boyer–Lindquist coordinates (see e.g. Refs. 36–38):

$$ds^2 = -A(dt)^2 - 2E\, dt\, d\varphi + \frac{\Sigma}{\Delta}(dr)^2 + \Sigma(d\theta)^2 + D\sin^2\theta(d\varphi)^2\,, \tag{34}$$

$$\begin{aligned} A &\equiv \frac{\Delta - a^2\sin^2\theta}{\Sigma}\,, \\ E &\equiv \frac{a\sin^2\theta(r^2 + a^2 - \Delta)}{\Sigma}\,, \\ D &\equiv \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\,, \end{aligned} \tag{35}$$

where $\Sigma \equiv r^2 + a^2 \cos^2\theta$, $\Delta \equiv r^2 + a^2 + e^2 - 2mr$. Let us recall that the Kerr–Newman metric (34) and (35) reduces to the Reissner–Nordström metric in the limiting case $a = 0$.

For the LL-brane embedding we will use the following ansatz:

$$X^0 \equiv t = \tau\,, \quad r = r(\tau)\,, \quad \theta = \sigma^1\,, \quad \varphi = \sigma^2 + \tilde{\varphi}(\tau)\,. \tag{36}$$

In this case the LL-brane equations of motion (23), (24) acquire the form:

$$\begin{gathered} -A + \frac{\Sigma}{\Delta}\dot{r}^2 + D\sin^2\theta\,\dot{\varphi}^2 - 2E\dot{\varphi} = 0\,, \\ -E + D\sin^2\theta\,\dot{\varphi} = 0\,, \quad \frac{d}{d\tau}(D\Sigma\sin^2\theta) = 0\,. \end{gathered} \tag{37}$$

Inserting the ansatz (36) into (37) the last Eq. (37) implies:

$$r(\tau) = r_0 = \text{const}\,, \tag{38}$$

whereas the second Eq. (37) yields:

$$\Delta(r_0) = 0\,, \quad \omega \equiv \dot{\varphi} = \frac{a}{r_0^2 + a^2} \tag{39}$$

Eqs. (38), (39) indicate that:

(i) the LL-brane automatically locates itself on the Kerr–Newman horizon $r = r_0$ — horizon "straddling" according to the terminology of Ref. 4;
(ii) the LL-brane rotates along with the same angular velocity ω as the Kerr–Newman horizon.

The first equation in (37) implies that $\dot{r}$ vanishes on-shell as:

$$\dot{r} \simeq \pm \frac{\Delta(r)}{r_0^2 + a^2}\bigg|_{r\to r_0}\,. \tag{40}$$

We will also need the explicit form of the last equation in (24) (using notations (35)):

$$\gamma_{ij} = \frac{1}{2a_0}\begin{pmatrix} \Sigma & 0 \\ 0 & D\sin^2\theta \end{pmatrix}\bigg|_{r=r_0,\theta=\sigma^1}\,. \tag{41}$$

Among the X^μ-equations of motion (26) only the X^0-equation yields additional information. Because of the embedding $X^0 = \tau$ it acquires the form of a time-evolution equation for the dynamical brane tension χ:

$$\partial_\tau \chi + \chi\big[\partial_\tau X^\nu \partial_\tau X^\lambda - \gamma^{ij}\partial_i X^\nu \partial_j X^\lambda\big]\Gamma^0_{\nu\lambda} = 0\,, \tag{42}$$

which, after taking into account (36), (38), (39) and the explicit expressions for the Kerr–Newman Christoffel connection coefficients (Ref. 36), reduces to:

$$\partial_\tau \chi + \chi 2\dot{r}\left[\Gamma^0_{0r} + \frac{a}{r_0^2 + a^2}\,\Gamma^0_{r\varphi}\right]_{r=r_0} = 0\,. \tag{43}$$

Singularity on the horizon of the Christoffel coefficients ($\sim \Delta^{-1}$) appearing in (43) is canceled by Δ in $\dot{r}$ (40) so that finally we obtain:

$$\partial_\tau \chi \pm \chi \frac{2(r_0 - m)}{r_0^2 + a^2} = 0\,, \quad \text{i.e.} \quad \chi = \chi_0 \exp\left\{\mp 2 \frac{(r_0 - m)}{r_0^2 + a^2}\,\tau\right\}. \tag{44}$$

Thus, we find "mass inflation/deflation" effect (according to the terminology of Refs. 21 and 22) on the Kerr–Newman horizon via the exponential time dependence of the dynamical LL-brane tension similar to the "mass inflation/deflation" effect with LL-branes in spherically symmetric gravitational backgrounds (Eq. (33)).

Now let us consider rotating cylindrical black hole background in $D = 4$:[39,40]

$$ds^2 = -A(dt)^2 - 2E\,dt\,d\varphi + \frac{(dr)^2}{\Delta} + D(d\varphi)^2 + \alpha^2 r^2 (dz)^2\,, \tag{45}$$

where

$$\begin{aligned} A &\equiv -\omega^2 r^2 + \gamma^2 \Delta\,, \quad & E &\equiv \gamma\omega r^2 - \frac{\gamma\omega}{\alpha^2}\Delta\,, \\ D &\equiv \gamma^2 r^2 - \frac{\omega^2}{\alpha^4}\Delta\,, \quad & \Delta &\equiv \alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2}\,. \end{aligned} \tag{46}$$

The physical meaning of the parameters involved is as follows: $\alpha^2 = -\frac{1}{3}\Lambda$, i.e. Λ must be negative cosmological constant; $b = 4m$ with m being the mass per unit length along the z-axis; $c^2 = 4\lambda^2$, where λ indicates the linear charge density along the z-axis.

The metric (46) possesses in general two horizons at $r = r_{(\pm)}$ where $\Delta\big(r_{(\pm)}\big) = 0$.

Let us note the useful identity which will play an important role in the sequel:

$$AD + E^2 = r^2 \left(\gamma^2 - \frac{\omega^2}{\alpha^2}\right)^2 \Delta\,. \tag{47}$$

For the LL-brane embedding we will use an ansatz similar to the Kerr–Newman case (36):

$$X^0 \equiv t = \tau\,, \quad r = r(\tau)\,, \quad z = \sigma^1\,, \quad \varphi = \sigma^2 + \tilde{\varphi}(\tau)\,. \tag{48}$$

Then the lightlike and Virasoro-like constraint equations of the LL-brane dynamics (24) in the background (45), (46) (the analogs of Eqs. (37) in the Kerr–Newman case):

$$-A + \frac{\dot{r}^2}{\Delta} + D\dot{\varphi}^2 - 2E\dot{\varphi} = 0\,, \quad -E + D\dot{\varphi} = 0\,, \quad \frac{d}{d\tau}(D\alpha^2 r^2) = 0 \tag{49}$$

imply:

$$r(\tau) = r_0 = \text{const}\,, \quad \Delta(r_0) = 0\,, \quad \dot{\varphi} = \frac{\omega}{\gamma}\,. \tag{50}$$

Thus, similarly to the Kerr–Newman case:

(i) the LL-brane automatically locates itself on one of the cylindrical black hole horizons at $r = r_0 = r_{(\pm)}$ (horizon "straddling");

(ii) the LL-brane rotates with angular velocity ω/γ along with the rotation of the cylindrical black hole horizon;

(iii) $\dot{r}$ vanishes on-shell as:

$$\dot{r} \simeq \pm \left| \frac{1}{\gamma}\left(\gamma^2 - \frac{\omega^2}{\alpha^2}\right) \right| \Delta(r) \Big|_{r \to r_{(\pm)}} \tag{51}$$

(the overall signs $\pm$ on the r.h.s. of (51) are *not* correlated with the indices $(\pm)$ labeling the outer/inner horizon).

Again in complete analogy with the Kerr–Newman case (Eqs. (42), (43)) the X^0-equation of motion (26) reduces to the following time-evolution equation for the pertinent dynamical brane tension:

$$\partial_\tau \chi + \chi 2\dot{r} \left[\Gamma^0_{0r} + \frac{E}{D} \Gamma^0_{r\varphi} \right]_{r=r_{(\pm)}} = 0\,, \tag{52}$$

where the Christoffel coefficients read:

$$\Gamma^0_{0r} = \frac{D\partial_r A + E\partial_r E}{2(AD + E^2)}\,, \quad \Gamma^0_{r\varphi} = \frac{D\partial_r E - E\partial_r D}{2(AD + E^2)} \tag{53}$$

with the functions A, D, E as in (45), (46). Taking into account (50), (51) and the identity (47) we obtain from (52), (53) exponential "inflation/deflation" of the LL-brane tension in rotating cylindrical black hole background:

$$\chi(\tau) = \chi_0 \exp\left\{ \mp \tau \left| \frac{1}{\gamma}\left(\gamma^2 - \frac{\omega^2}{\alpha^2}\right) \right| \partial_r \Delta(r) \Big|_{r=r_{(\pm)}} \right\} \tag{54}$$

(here again there is no correllation between the overall signs $\mp$ in the exponent with the indices $(\pm)$ labeling the outer/inner horizon).

4. Self-Consistent Wormhole Solutions via Lightlike Branes

Let us now consider a self-consistent bulk Einstein–Maxwell system (with a cosmological constant) free of electrically charged matter, coupled to a codimension one LL-brane:

$$S = \int d^D x \sqrt{-G} \left[\frac{R(G)}{16\pi} - \frac{\Lambda}{8\pi} - \frac{1}{4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \right] + S_{\rm LL}\,. \tag{55}$$

Here $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$ and $S_{\rm LL}$ is the same LL-brane worldvolume action as in (8). Thus, the LL-brane will serve as a gravitational source through its energy–momentum tensor (see Eq. (57) below). The pertinent Einstein–Maxwell equations of motion read:

$$\begin{aligned} &R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R + \Lambda G_{\mu\nu} = 8\pi \left(T^{(\rm EM)}_{\mu\nu} + T^{(\rm brane)}_{\mu\nu} \right), \\ &\partial_\nu \left(\sqrt{-G} G^{\mu\kappa} G^{\nu\lambda} \mathcal{F}_{\kappa\lambda} \right) = 0\,, \end{aligned} \tag{56}$$

where $T^{(\mathrm{EM})}_{\mu\nu} = \mathcal{F}_{\mu\kappa}\mathcal{F}_{\nu\lambda}G^{\kappa\lambda} - G_{\mu\nu}\frac{1}{4}\mathcal{F}_{\rho\kappa}\mathcal{F}_{\sigma\lambda}G^{\rho\sigma}G^{\kappa\lambda}$, and the LL-brane energy–momentum tensor is straightforwardly derived from (8):

$$T^{(\mathrm{brane})}_{\mu\nu} = -G_{\mu\kappa}G_{\nu\lambda}\int d^{p+1}\sigma\,\frac{\delta^{(D)}(x - X(\sigma))}{\sqrt{-G}}\,\chi\sqrt{-\gamma}\gamma^{ab}\partial_a X^\kappa\partial_b X^\lambda\,. \tag{57}$$

The equations of motion of the LL-brane have already been given in (24)–(26).

Using (57) we will now construct traversable *wormhole* solutions to the Einstein equations (56) which will combine the features of the Einstein–Rosen "bridge" (wormhole throat at horizon) and the feature "charge without charge" of Misner–Wheeler wormholes.[31] In doing this we will follow the standard procedure described in Ref. 23, but with the significant difference that in our case we will solve Einstein equations following from a self-consistent bulk gravity–matter system coupled to a LL-brane. In other words, the LL-brane will serve as a gravitational source of the wormhole by locating itself on its throat as a result of its consistent worldvolume dynamics (Eq. (32) above).

First we will consider the case with *spherical symmetry.* To this end let us take a spherically symmetric solution of (56) of the form (31) in the absence of the LL-brane (i.e. without $T^{(\mathrm{brane})}_{\mu\nu}$ on the r.h.s.), which possesses an (outer) event horizon at some $r = r_0$ (i.e. $A(r_0) = 0$ and $A(r) > 0$ for $r > r_0$). At this point we introduce the following modification of the metric (31):

$$\begin{aligned} ds^2 &= -\tilde{A}(\eta)(dt)^2 + \tilde{A}^{-1}(\eta)(d\eta)^2 \\ &\quad + (r_0 + |\eta|)^2 h_{ij}(\boldsymbol{\theta})d\theta^i\, d\theta^j\,, \\ \tilde{A}(\eta) &\equiv A(r_0 + |\eta|)\,, \end{aligned} \tag{58}$$

where $-\infty < \eta < \infty$. From now on the bulk space–time indices μ, ν will refer to (t, η, θ^i) instead of (t, r, θ^i). The new metric (58) represents two identical copies of the exterior region ($r > r_0$) of the spherically symmetric space–time with metric (31), which are sewed together along the horizon $r = r_0$. We will show that the new metric (58) is a solution of the full Einstein equations (56), *including* $T^{(\mathrm{brane})}_{\mu\nu}$ on the r.h.s. Here the newly introduced coordinate η will play the role of a radial-like coordinate normal w.r.t. the LL-brane located on the horizon, which interpolates between two copies of the exterior region of (31) (the two copies transform into each other under the "parity" transformation $\eta \to -\eta$).

Inserting in (57) the expressions for $X^\mu(\sigma)$ from (28) and (32) and taking into account (16), (21), (22) we get:

$$T^{\mu\nu}_{(\mathrm{brane})} = S^{\mu\nu}\,\delta(\eta) \tag{59}$$

with surface energy–momentum tensor:

$$S^{\mu\nu} \equiv -\frac{\chi}{(2a_0)^{p/2}}\left[-\partial_\tau X^\mu\partial_\tau X^\nu + \gamma^{ij}\partial_i X^\mu\partial_j X^\nu\right]_{t=\tau,\eta=0,\theta^i=\sigma^i}\,, \quad \partial_i \equiv \frac{\partial}{\partial\sigma^i}\,, \tag{60}$$

where again a_0 is the integration constant parameter appearing in the LL-brane dynamics (cf. Eq. (16)). Let us also note that unlike the case of test LL-brane moving in a spherically symmetric background (Eqs. (30) and (33)), the dynamical brane tension χ in Eq. (60) turns out to be *constant*. This is due to the fact that in the present context we have a discontinuity in the Christoffel connection coefficients across the LL-brane sitting on the horizon ($\eta = 0$). The problem in treating the geodesic LL-brane equations of motion (19), in particular — Eq. (30), can be resolved following the approach in Ref. 3 (see also the regularization approach in Ref. 41, App. A) by taking the mean value of the pertinent nonzero Christoffel coefficients across the discontinuity at $\eta = 0$. From the explicit form of Eq. (30) it is straightforward to conclude that the above mentioned mean values around $\eta = 0$ vanish since now ∂_r is replaced by $\partial/\partial\eta$, whereas the metric coefficients depend explicitly on $|\eta|$. Therefore, in the present case Eq. (30) is reduced to $\partial_\tau \chi = 0$.

Let us now separate in (56) explicitly the terms contributing to δ-function singularities (these are the terms containing second derivatives w.r.t. η, bearing in mind that the metric coefficients in (58) depend on $|\eta|$):

$$
\begin{aligned}
R_{\mu\nu} &\equiv \partial_\eta \Gamma^\eta_{\mu\nu} - \partial_\mu \partial_\nu \ln \sqrt{-G} + \text{nonsingular terms} \\
&= 8\pi \left(S_{\mu\nu} - \frac{1}{p} G_{\mu\nu} S^\lambda_\lambda \right) \delta(\eta) + \text{nonsingular terms} . \qquad (61)
\end{aligned}
$$

The only nonzero contribution to the δ-function singularities on both sides of Eq. (61) arises for $(\mu\nu) = (\eta\eta)$. In order to avoid coordinate singularity on the horizon it is more convenient to consider the mixed component version of the latter (with one contravariant and one covariant index):

$$
R^\eta_\eta = 8\pi \left(S^\eta_\eta - \frac{1}{p} S^\lambda_\lambda \right) \delta(\eta) + \text{nonsingular terms} \qquad (62)
$$

(in the special case of Schwarzschild geometry we can use the Eddington–Finkelstein coordinate system which is free of singularities on the horizon; see the Appendix). Evaluating the l.h.s. of (62) through the formula (recall $D = p + 2$):

$$
R^r_r = -\frac{1}{2} \frac{1}{r^{D-2}} \partial_r (r^{D-2} \partial_r A) \qquad (63)
$$

valid for any spherically symmetric metric of the form (31) and recalling $r = r_0 + |\eta|$, we obtain the following matching condition for the coefficients in front of the δ-functions on both sides of (62) (analog of Israel junction conditions[3,4]):

$$
\partial_\eta \tilde{A}\big|_{\eta \to +0} - \partial_\eta \tilde{A}\big|_{\eta \to -0} = -\frac{16\pi\chi}{(2a_0)^{p/2-1}} , \qquad (64)
$$

or, equivalently:

$$
\partial_r A|_{r=r_0} = -\frac{8\pi\chi}{(2a_0)^{p/2-1}} , \qquad (65)
$$

where we have used the explicit expression for the trace of the LL-brane energy–momentum tensor (60):

$$S^{\lambda}_{\lambda} = -\frac{p}{(2a_0)^{p/2-1}}\,\chi\,. \tag{66}$$

Equation (65) yields a relation between the parameters of the spherically symmetric outer regions of "vacuum" solution (31) of Einstein equations (56) and the dynamical tension of the LL-brane sitting at the (outer) horizon.

As an explicit example let us take (31) to be the standard $D = 4$ Reissner–Nordström metric, i.e. $A(r) = 1 - \frac{2m}{r} + \frac{e^2}{r^2}$. Then Eq. (64) yields the following relation between the Reissner–Nordström parameters and the dynamical LL-brane tension:

$$4\pi\chi r_0^2 + r_0 - m = 0 \quad \text{where} \quad r_0 = m + \sqrt{m^2 - e^2}\,. \tag{67}$$

Equation (67) indicates that the dynamical brane tension must be *negative*. Equation (67) reduces to a cubic equation for the Reissner–Nordström mass m as function of $|\chi|$:

$$(16\pi|\chi|m - 1)(m^2 - e^2) + 16\pi^2\chi^2 e^4 = 0\,. \tag{68}$$

In the special case of Schwarzschild wormhole ($e^2 = 0$) the Schwarzschild mass becomes:

$$m = \frac{1}{16\pi|\chi|}\,. \tag{69}$$

The particular case of Schwarzschild wormhole $\left(\text{with } \tilde{A}(\eta) = 1 - \frac{2m}{2m+|\eta|}\right)$ constructed above is the proper consistent realization of the Einstein–Rosen "bridge".[27] We refer to the Appendix, where it is explained how the present formalism involving a LL-brane as wormhole source positioned at the wormhole throat resolves certain inconsistency in the original treatment of the Einstein–Rosen "bridge."

Let us observe that for large values of the LL-brane tension $|\chi|$, the Reissner–Nordström (Schwarzschild) mass m is very small. In particular, $m \ll M_{\rm Pl}$ for $|\chi| > M^3_{\rm Pl}$ ($M_{\rm Pl}$ being the Planck mass). On the other hand, for small values of the LL-brane tension $|\chi|$ Eq. (67) implies that the Reissner–Nordström geometry of the wormhole must be near extremal ($m^2 \simeq e^2$).

Now we will apply the above formalism to construct a rotating cylindrically symmetric wormhole in $D = 4$. Namely, we introduce the following modification of the cylindrically symmetric rotating black hole metric (45) and (46) (cf. (58) above):

$$ds^2 = -\tilde{A}(dt)^2 - 2\tilde{E}dt\,d\varphi + \frac{(d\eta)^2}{\tilde{\Delta}} + \tilde{D}(d\varphi)^2 + \alpha^2(r_{(+)} + |\eta|)^2(dz)^2\,, \tag{70}$$

where

$$\begin{aligned} \tilde{A}(\eta) &= A(r_{(+)} + |\eta|)\,, \quad \tilde{D}(\eta) = D(r_{(+)} + |\eta|)\,,\\ \tilde{E}(\eta) &= E(r_{(+)} + |\eta|)\,, \quad \tilde{\Delta}(\eta) = \Delta(r_{(+)} + |\eta|)\,, \end{aligned} \tag{71}$$

with A, D, E, Δ the same as in (46), and $r_{(+)}$ indicates the outer horizon of (45). From now on the bulk space–time indices μ, ν will refer to (t, η, z, φ) (instead of (t, r, z, φ)).

The metric (70), (71) represents two identical copies of the exterior region ($r > r_{(+)}$) of the cylindrically symmetric rotating black hole space–time with metric (45), which are sewed together along the outer horizon $r = r_{(+)}$. The newly introduced coordinate η ($-\infty < \eta < \infty$) will play the role of a planar radial-like coordinate normal w.r.t. the LL-brane located on the horizon, which interpolates between two copies of the exterior region of (31) (the two copies transform into each other under the "parity" transformation $\eta \to -\eta$).

In the present case the LL-brane energy–momentum tensor (57) has again the form (59) with surface energy–momentum tensor (cf. Eq. (60) above; now we have $D = p + 2 = 4$):

$$S^{\mu\nu} = -\frac{\chi}{2a_0} \frac{|\gamma|}{|\gamma^2 - \omega^2/\alpha^2|} \times \left[-\partial_\tau X^\mu \partial_\tau X^\nu + \gamma^{ij} \partial_i X^\mu \partial_j X^\nu\right]_{t=\tau, \eta=0, z=\sigma^1, \varphi=\sigma^2+\tau\omega/\gamma}, \tag{72}$$

where (48) and (50) are taken into account. Here once again the dynamical LL-brane tension χ turns out to be *constant* unlike the exponential "inflation/deflation" (54) of the tension of test LL-brane moving in a fixed cylindrical black hole background. The proof is completely analogous to the one given above for the spherically symmetric case.

As in the spherically symmetric case we separate in Einstein equations (56) explicitly the terms contributing to δ-function singularities on the LL-brane world-volume (obtaining Eqs. (61)), where again only the $(\mu\nu) = (\eta\eta)$ equation contains nonzero δ-function contributions, i.e. arriving at Eq. (62). In the present case of cylindrically symmetric geometry (70), (71), Eq. (62) yields (taking into account the explicit form (72) of the LL-brane surface energy–momentum tensor):

$$-\frac{1}{2}\partial_\eta^2 \Delta(r_{(+)} + |\eta|) = 8\pi\chi \frac{|\gamma|}{|\gamma^2 - \omega^2/\alpha^2|} \delta(\eta) + \text{nonsingular terms}, \tag{73}$$

i.e.

$$\partial_\eta \Delta(r_{(+)} + |\eta|)\big|_{\eta\to+0} - \partial_\eta \Delta(r_{(+)} + |\eta|)\big|_{\eta\to-0} = -16\pi\chi \frac{|\gamma|}{|\gamma^2 - \omega^2/\alpha^2|}, \tag{74}$$

or, equivalently:

$$\partial_r \Delta(r)|_{r=r_{(+)}} = -8\pi\chi \frac{|\gamma|}{|\gamma^2 - \omega^2/\alpha^2|}. \tag{75}$$

Equation (75) provides a relation between the parameters α, b, c of the cylindrical rotating wormhole and the dynamical tension χ of the wormhole-generating LL-brane. Here by construction the l.h.s. of (75) is strictly positive since $r_{(+)}$ is the outer horizon of the original metric (70), (71), therefore, again as in the spherically symmetric case the dynamical LL-brane tension χ at the wormhole throat turns out to be *negative*.

5. Traversability Considerations

Let us now briefly consider the dynamics of a test point particle moving in the gravitational field of the cylindrically symmetric rotating wormhole constructed in the previous section. The analysis follows the lines of the standard procedure (see, e.g. Ref. 42). The pertinent reparametrization invariant test-particle action reads:

$$S_{\text{particle}} = \frac{1}{2}\int d\lambda \left[\frac{1}{e}\dot{x}^\mu \dot{x}^\nu G_{\mu\nu}(x) - em^2\right], \tag{76}$$

where $G_{\mu\nu}(x) \equiv G_{\mu\nu}(t,\eta,z,\varphi)$ is given by (70), (71) and e denotes the "einbein." Variation w.r.t. e yields the "mass-shell" constraint equation:

$$-\tilde{A}\dot{t}^2 - 2\tilde{E}\dot{t}\dot{\varphi} + \tilde{D}\dot{\varphi}^2 + \alpha^2(r_{(+)} + |\eta|)^2 \dot{z}^2 + \frac{1}{\tilde{\Delta}}\dot{\eta}^2 + e^2 m^2 = 0\,, \tag{77}$$

with $\tilde{A}$, $\tilde{D}$, $\tilde{E}$, $\tilde{\Delta}$ defined through (71) and (46). We have also three Noether conserved quantities — energy $\mathcal{E}$, axial angular momentum $\mathcal{J}$ and momentum P_z along the z-axis:

$$\mathcal{E} = \frac{1}{e}(\tilde{A}\dot{t} + \tilde{E}\dot{\varphi})\,, \quad \mathcal{J} = \frac{1}{e}(-\tilde{E}\dot{t} + \tilde{D}\dot{\varphi})\,, \quad P_z = \frac{1}{e}\alpha^2\big(r_{(+)} + |\eta|\big)^2 \dot{z}\,. \tag{78}$$

Solving (78) for $\dot{t}$, $\dot{\varphi}$, $\dot{z}$, substituting into (77) and employing the particle's proper-time s instead of the generic evolution parameter λ (the relation in the present case being given by $ds/d\lambda = em$) we obtain the equation for the particle motion along η — normal w.r.t. the wormhole throat:

$$\eta'^2 + \mathcal{V}_{\text{eff}}(|\eta|) = \mathcal{E}_{\text{eff}}\,, \tag{79}$$

where

$$\mathcal{V}_{\text{eff}}(|\eta|) \equiv \tilde{\Delta}(|\eta|)\left[1 + \frac{1}{m^2(r_{(+)} + |\eta|)^2}\left(\frac{P_z^2}{\alpha^2} + \frac{(\gamma\mathcal{J} - \mathcal{E}\omega/\alpha^2)^2}{(\gamma^2 - \omega^2/\alpha^2)^2}\right)\right], \tag{80}$$

$$\mathcal{E}_{\text{eff}} \equiv \frac{(\gamma\mathcal{E} - \omega\mathcal{J})^2}{m^2(\gamma^2 - \omega^2/\alpha^2)^2}\,. \tag{81}$$

The "effective potential" $\mathcal{V}_{\text{eff}}$ (80) is strictly positive for each $\eta \neq 0$ and growing with the following behavior for small η (i.e. around the wormhole throat) and large η, respectively:

$$\mathcal{V}_{\text{eff}}(|\eta|)\begin{cases} \simeq \text{const}\,|\eta| & \text{for } \eta \to 0\,, \\ \simeq \text{const}\,\eta^2 & \text{for } \eta \to \pm\infty\,. \end{cases} \tag{82}$$

Therefore, there is a whole range of values of the "effective energy" $\mathcal{E}_{\text{eff}}$ (81) for which the test particle periodically traverses the wormhole between the "turning points" $\pm\eta_{\text{turning}}$ ($\mathcal{V}_{\text{eff}}(|\pm\eta_{\text{turning}}|) = \mathcal{E}_{\text{eff}}$) within a *finite* amount of its proper time s.

On the other hand, for a static observer on either side of the wormhole (which by construction is a copy of the exterior region of a black hole beyond the outer horizon) the throat looks the same as a black hole horizon, therefore, it would take an infinite amount of the "laboratory" time t for a test particle to reach the wormhole throat. Thus, when we say that we have constructed *traversable* wormholes via LL-branes, we have in mind traversability w.r.t. *proper time* of travellers.

6. Conclusions

In the present paper we have provided a systematic general scheme to construct self-consistent spherically symmetric or rotating cylindrical wormhole solutions via LL-branes, such that the latter occupy the wormhole throats and match together two copies of exterior regions of spherically symmetric or rotating cylindrical black holes (the regions beyond the outer horizons).

As a particular case, the matching of two exterior regions of Schwarzschild space–time at the horizon surface $r = 2m$ through a LL-brane is indeed the self-consistent realization of the original Einstein–Rosen "bridge," namely, it requires the presence of a LL-brane at $r = 2m$ — a feature not recognized in the original Einstein–Rosen work[27] (see Appendix).

The main result here is the construction of self-consistent rotating wormholes with rotating LL-brane as their sources sitting at the wormhole throats. The surface tension of the LL-brane is an additional brane degree of freedom, which assumes negative values on-shell in all cases — both for spherically symmetric as well as for rotating cylindrical wormholes, but it can be of arbitrary small magnitude. The latter means that the LL-brane represents an "exotic" matter due to violation of the null-energy conditions, which is in accordance with the general wormhole arguments.[23]

The class of spherically symmetric and rotating cylindrical wormhole solutions produced by LL-brane constructed above combine the features of the original Einstein–Rosen "bridge" manifold[27] (wormhole throat located at horizon) with the feature "charge without charge" of Misner–Wheeler wormholes.[31] There exist several other types of physically interesting wormhole solutions in the literature generated by different types of matter and without horizons. For a recent discussion, see Ref. 43 and references therein.

The geodesic equations for test particles traversing the throats of the presently constructed wormholes have been briefly studied. We have found that it requires a *finite proper time* for a traveling observer to pass from one side of the wormhole to the other, so that "traversability" for the presently constructed wormholes via LL-branes is understood as traversability w.r.t. the proper time of traveling observers.

Appendix A. The Original Einstein–Rosen "Bridge" Needs a Lightlike Brane for Consistency

Here we briefly examine the first explicit wormhole construction proposed by Einstein and Rosen[27] which is usually referred to as "Einstein–Rosen bridge." As we will see in what follows, the Einstein–Rosen "bridge" solution in terms of the original coordinates introduced in Ref. 27 (Eq. (A.1) below) *does not* satisfy the vacuum Einstein equations due to an ill-defined δ-function contribution at the throat appearing on the r.h.s. — a would-be "thin shell" matter energy–momentum tensor (see Eq. (A.6) below). The fully consistent formulation of the original Einstein–Rosen "bridge," namely, two identical copies of the exterior Schwarzschild space–time region matched along the horizon must include a gravity coupling to a LL-brane. In fact, the Einstein–Rosen "bridge" wormhole is a particular case ($e = 0$) of our construction of Reissner–Nordström wormhole via LL-brane presented above in Sec. 4 (cf. Eqs. (64)–(69)). Here we will study separately the Einstein–Rosen "bridge" construction because of its historic importance.

Let us start with the coordinate system proposed in Ref. 27, which is obtained from the original Schwarzschild coordinates by defining $u^2 = r - 2m$, so that the Schwarzschild metric becomes:

$$ds^2 = -\frac{u^2}{u^2 + 2m}(dt)^2 + 4(u^2 + 2m)(du)^2 + (u^2 + 2m)^2((d\theta)^2 + \sin^2\theta(d\varphi)^2)\,. \tag{A.1}$$

Then Einstein and Rosen "double" the exterior Schwarzschild space–time region ($r > 2m$) by letting the new coordinate u to vary between $-\infty$ and $+\infty$ (i.e. we have the same $r \geq 2m$ for $\pm u$). The two Schwarzschild exterior space–time regions must be matched at the horizon $u = 0$.

At this point let us note that the notion of "Einstein–Rosen bridge" in e.g. Ref. 28, which uses the Kruskal–Szekeres manifold, is not equivalent to the original construction in Ref. 27, i.e. two identical copies of the exterior Schwarzschild space–time region matched along the horizon. The two regions in Kruskal–Szekeres space–time corresponding to the outer Schwarzschild space–time region ($r > 2m$) and labeled (I) and (III) in Ref. 28 are generally *disconnected* and share only a two-sphere (the angular part) as a common border ($U = 0, V = 0$ in Kruskal–Szekeres coordinates), whereas in the original Einstein–Rosen "bridge" construction the boundary between the two identical copies of the outer four-dimensional Schwarzschild space–time region ($r > 2m$) is a three-dimensional hypersurface ($r = 2m$).

In our wormhole construction above (Sec. 4) we have used a different new coordinate $\eta \in (-\infty, +\infty)$ to describe the two copies of the exterior (beyond the outer horizon) space–time regions. In the Schwarzschild case we have $|\eta| = r - 2m$ and, accordingly, the Schwarzschild metric describing both copies becomes:

$$ds^2 = -\frac{|\eta|}{|\eta| + 2m}(dt)^2 + \frac{|\eta| + 2m}{|\eta|}(d\eta)^2 + (|\eta| + 2m)^2((d\theta)^2 + \sin^2\theta(d\varphi)^2) \tag{A.2}$$

(one can use instead the Eddington–Finkelstein coordinate system; see below). Due to the nonsmooth dependence of the metric (A.2) on η via $|\eta|$ it is obvious that the terms in $R_{\mu\nu}$ containing second order derivative w.r.t. η will generate $\delta(\eta)$-terms on the l.h.s. of the pertinent Einstein equations. It is precisely due to the gravity coupling to LL-brane, that the corresponding LL-brane surface stress–energy tensor on the r.h.s. of Einstein equations matches the delta-function contributions on the l.h.s. In particular, calculating the scalar curvature of the metric (A.2) we obtain the well-defined nonzero distributional result:

$$R = -\frac{1}{m}\delta(\eta)\,. \tag{A.3}$$

The relation between both metrics (A.1) and (A.2) is a nonsmooth coordinate transformation:

$$u = \begin{cases} \sqrt{\eta} & \text{for } \eta \geq 0\,, \\ -\sqrt{-\eta} & \text{for } \eta \leq 0\,, \end{cases} \quad \text{i.e.} \quad u^2 = |\eta|\,. \tag{A.4}$$

Then the question arises as to whether one can see the presence of the LL-brane also in the Einstein–Rosen coordinates. The answer is "yes," but with the important disclaimer that the Einstein–Rosen coordinate u is *not appropriate* to describe the "bridge" at the throat $u = 0$ as it leads to an ill-defined δ-function singularity (Eq. (A.6) below).

To this end let us consider the Levi-Civita identity (see, e.g. Ref. 44):

$$R^0_0 = -\frac{1}{\sqrt{-g_{00}}}\,\nabla^2\big(\sqrt{-g_{00}}\big) \tag{A.5}$$

valid for any metric of the form $ds^2 = g_{00}(r)(dt)^2 + h_{ij}(r,\theta,\varphi)dx^i\,dx^j$ and where ∇^2 is the three-dimensional Laplace–Beltrami operator $\nabla^2 = \frac{1}{\sqrt{h}}\frac{\partial}{\partial x^i}\big(\sqrt{h}h^{ij}\frac{\partial}{\partial x^j}\big)$. The Einstein–Rosen metric (A.1) solves $R^0_0 = 0$ for all $u \neq 0$. However, since $\sqrt{-g_{00}} \sim |u|$ as $u \to 0$ and since $\frac{\partial^2}{\partial u^2}|u| = 2\delta(u)$, Eq. (A.5) tells us that:

$$R^0_0 \sim \frac{1}{|u|}\delta(u) \sim \delta(u^2)\,, \tag{A.6}$$

and similarly for the scalar curvature $R \sim \frac{1}{|u|}\delta(u) \sim \delta(u^2)$. From (A.6) we conclude that:

(i) The explicit presence of matter on the throat is missing in the original formulation[27] of Einstein–Rosen "bridge."

(ii) The coordinate u in (A.1) is *inadequate* for description of the original Einstein–Rosen "bridge" at the throat due to the *ill-definiteness* of the r.h.s. in (A.6).

(iii) One should use instead the coordinate η as in (A.2) (or as in (A.8), (A.9) below), which provides the consistent construction of the original Einstein–Rosen "bridge" manifold as a spherically symmetric wormhole with Schwarzschild geometry produced via lightlike brane sitting at its throat in a self-consistent formulation, namely, solving Einstein equations with a surface stress–energy tensor of the lightlike brane derived from a well-defined world-volume brane action. Moreover, the mass parameter m of the Einstein–Rosen "bridge" is not a free parameter but rather is a function of the dynamical LL-brane tension (Eq. (69)).

Let us also describe the construction of Einstein–Rosen "bridge" wormhole using the Eddington–Finkelstein coordinates for the Schwarzschild metric[45,46] (see also Ref. 28):

$$ds^2 = -A(r)(dv)^2 + 2dv\,dr + r^2[(d\theta)^2 + \sin^2\theta(d\varphi)^2]\,, \quad A(r) = 1 - \frac{2m}{r}\,. \tag{A.7}$$

The advantage of the metric (A.7) over the metric in standard Schwarzschild coordinates is that both (A.7) as well as the corresponding Christoffel coefficients *do not* exibit coordinate singularities on the horizon ($r = 2m$).

Let us introduce the following modification of (A.7) (cf. (A.2) above):

$$ds^2 = -\tilde{A}(\eta)(dv)^2 + 2dv\,d\eta + \tilde{r}^2(\eta)[(d\theta)^2 + \sin^2\theta(d\varphi)^2]\,, \tag{A.8}$$

where

$$\tilde{A}(\eta) = A(2m + |\eta|) = \frac{|\eta|}{|\eta| + 2m}\,, \quad \tilde{r}(\eta) = 2m + |\eta|\,. \tag{A.9}$$

The metric describes two identical copies of Schwarzschild *exterior* region ($r > 2m$) in terms of the Eddington–Finkelstein coordinates, which correspond to $\eta > 0$ and $\eta < 0$, respectively, and which are "glued" together at the horizon $\eta = 0$ (i.e. $r = 2m$), where the latter will serve as a throat of the overall wormhole solution.

We will show that the metric (A.8), (A.9) is a self-consistent solution of Einstein equations:

$$R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R = 8\pi T^{(\text{brane})}_{\mu\nu} \tag{A.10}$$

derived from the action describing bulk ($D = 4$) gravity coupled to a LL-brane:

$$S = \int d^4x\,\sqrt{-G}\,\frac{R(G)}{16\pi} + S_{\text{LL}}\,, \tag{A.11}$$

where S_{LL} is the LL-brane worldvolume action (8) with $p = 2$.

Using as above the simplest nontrivial ansatz for the LL-brane embedding coordinates $X^\mu \equiv (v, \eta, \theta, \varphi) = X^\mu(\sigma)$:

$$v = \tau \equiv \sigma^0\,, \quad \eta = \eta(\tau)\,, \quad \theta^1 \equiv \theta = \sigma^1\,, \quad \theta^2 \equiv \varphi = \sigma^2\,, \tag{A.12}$$

the pertinent LL-brane equations of motion yield (in complete analogy with (29)–(32)):

$$\eta(\tau) = 0\,, \quad \partial_\tau\chi + \chi\left[\frac{1}{2}\partial_\eta\tilde{A} + 2a_0\partial_\eta \ln\tilde{r}^2\right]_{\eta=0} = 0\,. \tag{A.13}$$

As above, the first Eq. (A.13) (horizon "straddling" by the LL-brane) is obtained from the constraint equations (24), whereas the second Eq. (A.13) results from the geodesic LL-brane equation for $X^0 \equiv v$ (26) due to the embedding (A.12). Here again as in Sec. 4, the problem with the discontinuity in the Christoffel coefficients accross the horizon ($\eta = 0$) is resolved following the approach in Ref. 3 (see also the regularization approach in Ref. 41, App. A), i.e. we need to take the mean value w.r.t. $\eta = 0$ yielding zero (since both $\tilde{A}$ and $\tilde{r}$ depend on $|\eta|$). Therefore, once again as in Sec. 4 we find that the dynamical LL-brane tension χ turns into an integration constant on-shell.

Taking into account (A.12), (A.13) the LL-brane energy–momentum tensor (57) derived from the worldvolume action (8) becomes (cf. Eq. (60)):

$$T^{\mu\nu}_{(\mathrm{brane})} = S^{\mu\nu}\,\delta(\eta)\,,$$
$$S^{\mu\nu} = \frac{\chi}{2a_0}[\partial_\tau X^\mu \partial_\tau X^\nu - 2a_0 G^{ij}\partial_i X^\mu \partial_j X^\nu]_{v=\tau,\eta=0,\theta^i=\sigma^i}\,, \tag{A.14}$$

where G^{ij} is the inverse metric in the (θ, φ) subspace and a_0 indicates the integration constant parameter arising in the LL-brane worldvolume dynamics (Eq. (16)).

Now we turn to the Einstein equations (A.10) where again as in Sec. 4 (cf. Eqs. (61)) we explicitly separate the terms contributing to δ-function singularities on the l.h.s.:

$$R_{\mu\nu} \equiv \partial_\eta \Gamma^\eta_{\mu\nu} - \partial_\mu\partial_\nu \ln\sqrt{-G} + \text{nonsingular terms}$$
$$= 8\pi\left(S_{\mu\nu} - \frac{1}{2}G_{\mu\nu}S^\lambda_\lambda\right)\delta(\eta)\,. \tag{A.15}$$

Using the explicit expressions:

$$\Gamma^\eta_{vv} = \frac{1}{2}\tilde{A}\partial_\eta\tilde{A}\,, \quad \Gamma^\eta_{v\eta} = -\frac{1}{2}\partial_\eta\tilde{A}\,, \quad \Gamma^\eta_{\eta\eta} = 0\,, \quad \sqrt{-G} = \tilde{r}^2 \tag{A.16}$$

with $\tilde{A}$ and $\tilde{r}$ as in (A.9), it is straightforward to check that nonzero δ-function contributions in $R_{\mu\nu}$ appear for $(\mu\nu) = (v\eta)$ and $(\mu\nu) = (\eta\eta)$ only. Using also the expressions $S_{\eta\eta} = \frac{1}{2a_0}\chi$ and $S^\lambda_\lambda = -2\chi$ (cf. Eq. (66)) the Einstein equations (A.15) yield for $(\mu\nu) = (v\eta)$ and $(\mu\nu) = (\eta\eta)$ the following matchings of the coefficients in front of the δ-functions, respectively:

$$m = \frac{1}{16\pi|\chi|}\,, \qquad m = \frac{a_0}{2\pi|\chi|}\,. \tag{A.17}$$

where as above the LL-brane dynamical tension must be negative. Consistency between the two relations (A.17) fixes the value $a_0 = 1/8$ for the integration constant a_0.

Thus, we recover the same expression for the Schwarzschild mass m of the Einstein–Rosen "bridge" wormhole (as function of the dynamical LL-brane tension) in the Eddington–Finkelstein coordinates (first Eq. (A.17)) as in the standard Schwarzschild coordinates (Eq. (69)). Moreover, unlike the previous treatment with the standard Schwarzschild coordinates, there are no coordinate singularities in the Christoffel coefficients (A.16), so when employing Eddington–Finkelstein coordinates there is no need to use mixed indices (one covariant and one contravariant) in the Einstein equations unlike (62).

In conclusion let us note that for the scalar curvature of the Einstein–Rosen "bridge" wormhole metric in Eddington–Finkelstein coordinates (A.8), (A.9) we obtain the same well-defined nonzero distributional result as in the case with ordinary Schwarzschild coordinates (A.3):

$$R = -\frac{1}{m}\,\delta(\eta)\,. \tag{A.18}$$

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